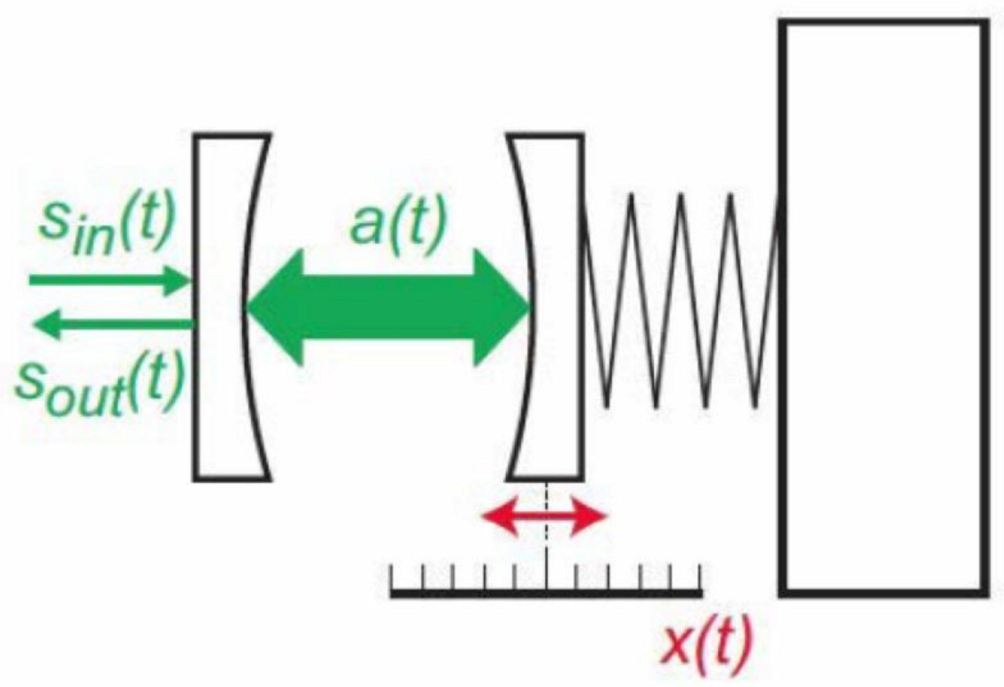
Hamiltonian description



$$\hat{F} = \hbar G n$$

measurement leads to a radiation pressure backaction

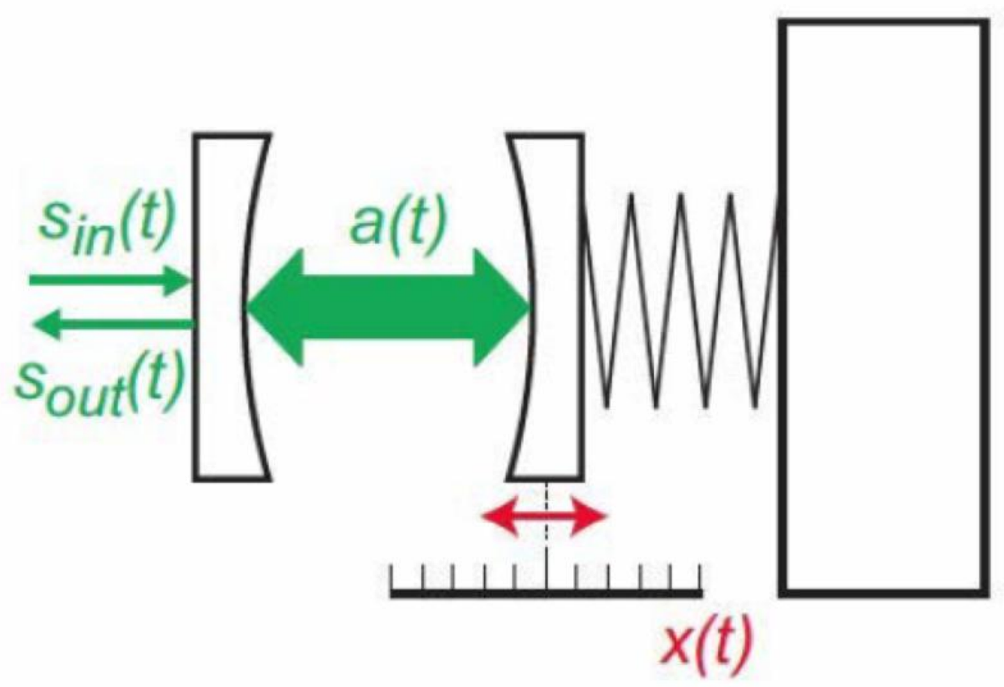
$$\hat{H} = \hbar\omega_0 \hat{a}^\dagger \hat{a} + \hbar\Omega_{\mathrm{m}} \hat{b}^\dagger \hat{b} + \hbar G \hat{a}^\dagger \hat{a} \hat{x}$$

Radiation pressure Force: $F_{\mathrm{RP}} = \frac{P}{\hbar\omega} 2\hbar\kappa = \bar{n}_{\mathrm{p}} \hbar G ; \hat{F}_{\mathrm{RP}} = \hbar G \hat{n}$

Force can be derived from a Hamiltonian: $\hat{H}_{\mathrm{int}} = (\hbar G \hat{n})\hat{x} = \hbar g_0 \hat{a}^\dagger \hat{a} (\hat{b}^\dagger + \hat{b})$

$$\hat{H}_{\mathrm{int}} = \hbar G x_{\mathrm{zpf}} \hat{a}^\dagger \hat{a} (\hat{b} + \hat{b}^\dagger) \qquad \hat{x} = x_{\mathrm{zpf}} (\hat{b} + \hat{b}^\dagger) \qquad x_{\mathrm{zpf}} = \sqrt{\frac{\hbar}{2m\Omega_{\mathrm{m}}}}$$

Hamiltonian description



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Hamiltonian description

$$\hat{H} = \hbar\omega_0 \hat{a}^\dagger \hat{a} + \hbar\Omega_m \hat{b}^\dagger \hat{b} + \hbar G \hat{a}^\dagger \hat{a} \hat{x}$$

Note: Hamiltonian is singly QND Hamiltonian for photon number

One can derive the equations of motion for the operators (optics and mechanics)

Optical frequency shift

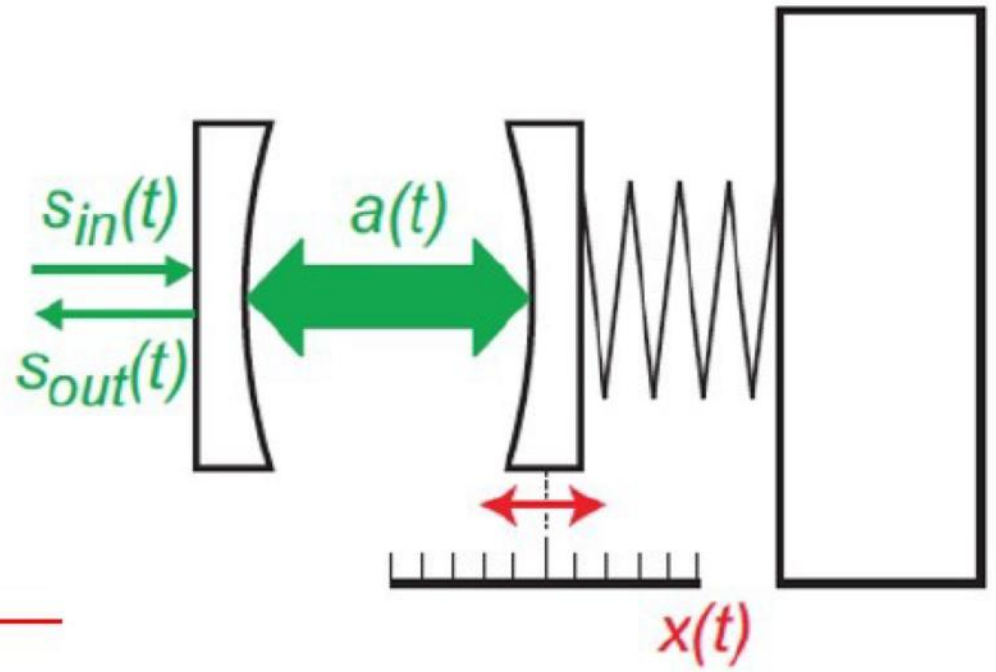
$$\dot{\hat{a}} = i\left[H, \hat{a}\right]/\hbar = i\left(\omega_c + G\hat{x}\right)\hat{a}$$

Radiation pressure force

$$\hat{F}_{rp} = i\left[H_{int}, p\right]/\hbar = \hbar G a^\dagger a$$

From Hamiltonian formulation one recovers the classical equation of motion (without the respective damping terms)

$$\hat{H}_{int} = \hbar G x_{zpm} \hat{a}^\dagger \hat{a}\left(\hat{b}^\dagger + \hat{b}\right)$$



g0 is the vacuum optomechanical coupling rate

$$g_0 = G\sqrt{\frac{\hbar}{2m\Omega_m}}$$

Hamiltonian description

$$\hat{H} = \hbar\omega_0 \hat{a}^\dagger \hat{a} + \hbar\Omega_m \hat{b}^\dagger \hat{b} + \hbar G \hat{a}^\dagger \hat{a} \hat{x}$$

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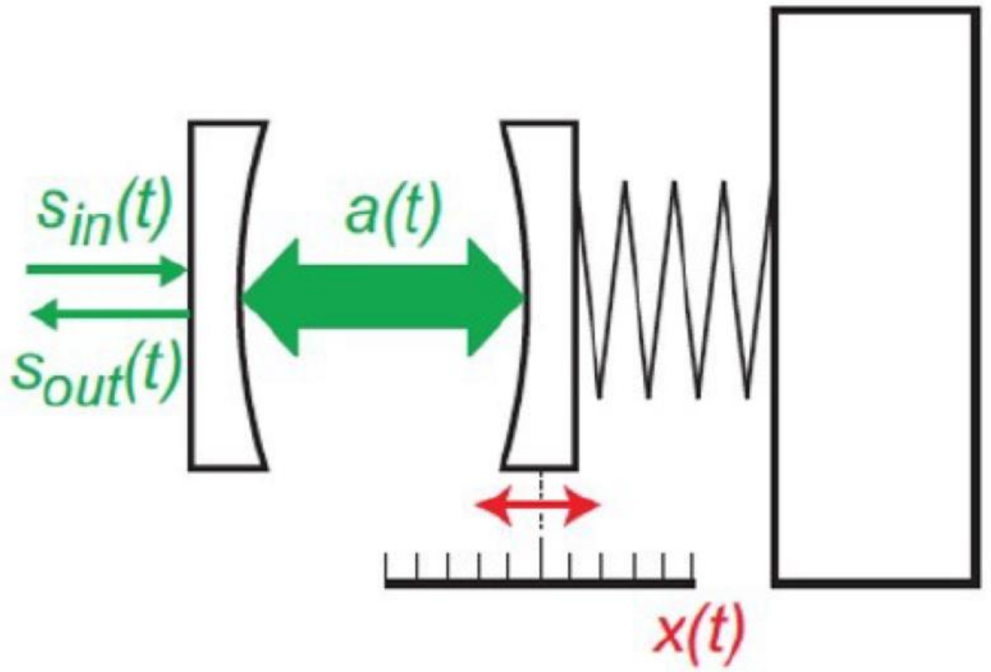

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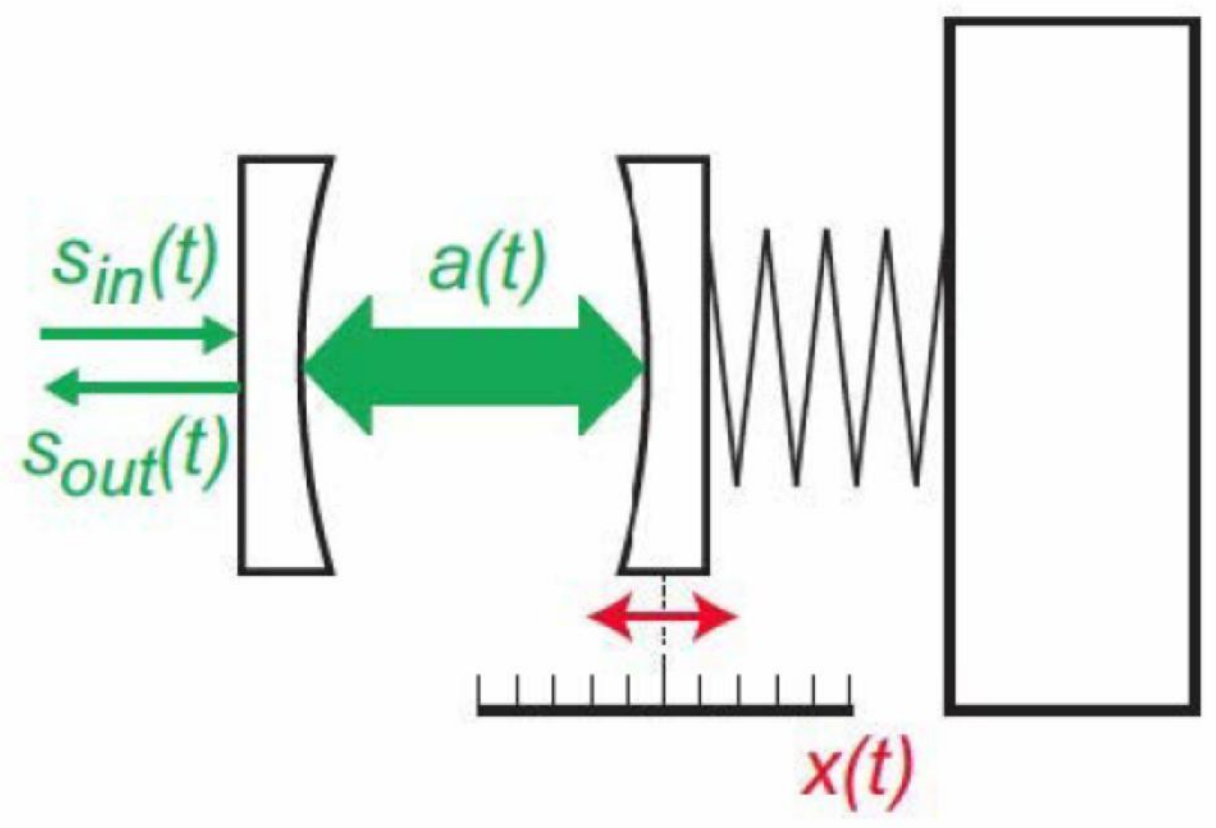


$$\hat{H} = \hbar\omega_0 \hat{a}^\dagger \hat{a} + \hbar\Omega_{\mathrm{m}} \hat{b}^\dagger \hat{b} + \underbrace{\hbar G \hat{a}^\dagger \hat{a} \hat{x}}_{\hat{\mathrm{H}}_{\mathrm{int}}}$$

Linearization around the driven cavity

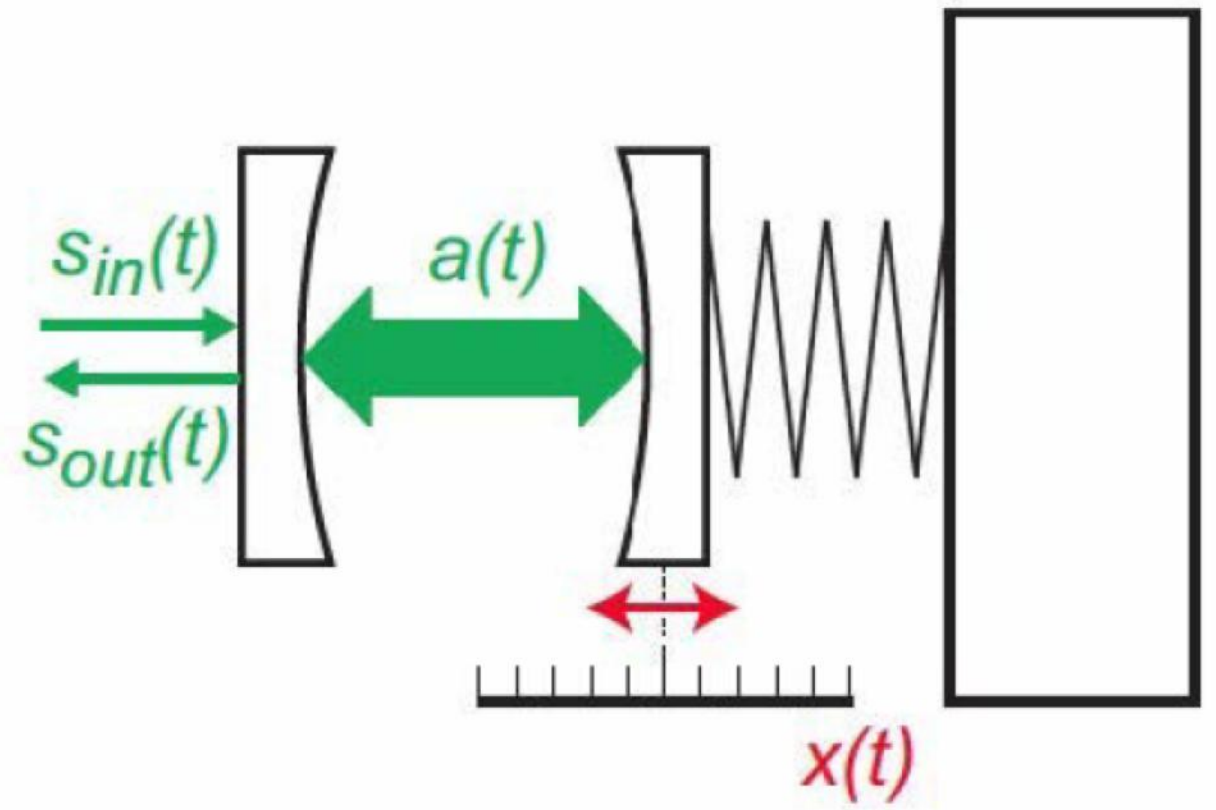
$$\hat{a} = \bar{a} + \delta\hat{a} \qquad \bar{a} = \sqrt{\bar{n}_{\mathrm{p}}}$$

$$\hat{x} = \bar{x} + x_{\mathrm{zpf}}(\delta\hat{b} + \delta\hat{b}^\dagger)$$

Quantum theory of optomechanical cooling:

I. Wilson-Rae, Nooshi, Zwerger, Kippenberg, PRL 99, 093901 (2007)
J. Dobrindt, Wilson-Rae, Kippenberg, PRL, 101, 263602 (2008)
F. Marquardt, Chen, Clerk, Girvin, PRL 99, 093902 (2007)

Hamiltonian description



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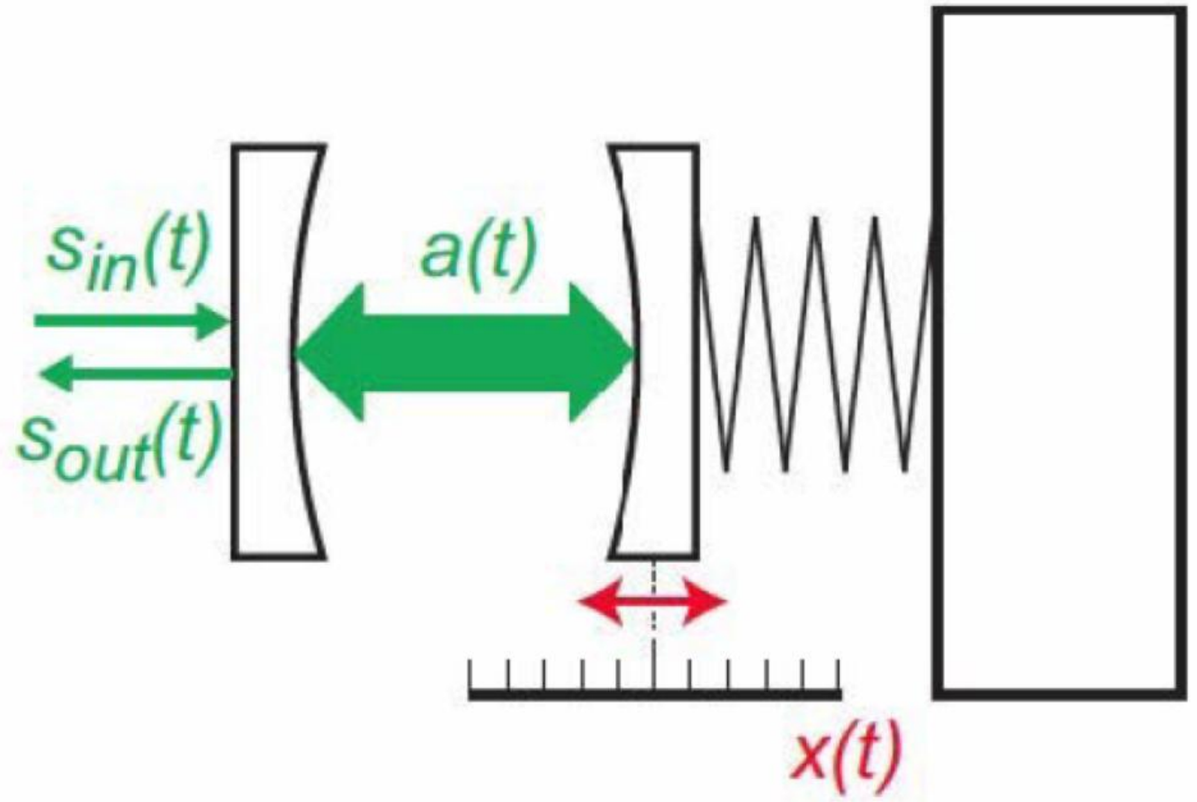


$$\hat{H} = \hbar\Delta\delta\hat{a}^\dagger\delta\hat{a} + \hbar\Omega_\mathrm{m}\delta\hat{b}^\dagger\delta\hat{b} + \hbar G x_\mathrm{zpf}\bar{a}(\delta\hat{a} + \delta\hat{a}^\dagger)(\delta\hat{b} + \delta\hat{b}^\dagger)$$

Resolved sideband regime: $\Delta = -\Omega_\mathrm{m}$

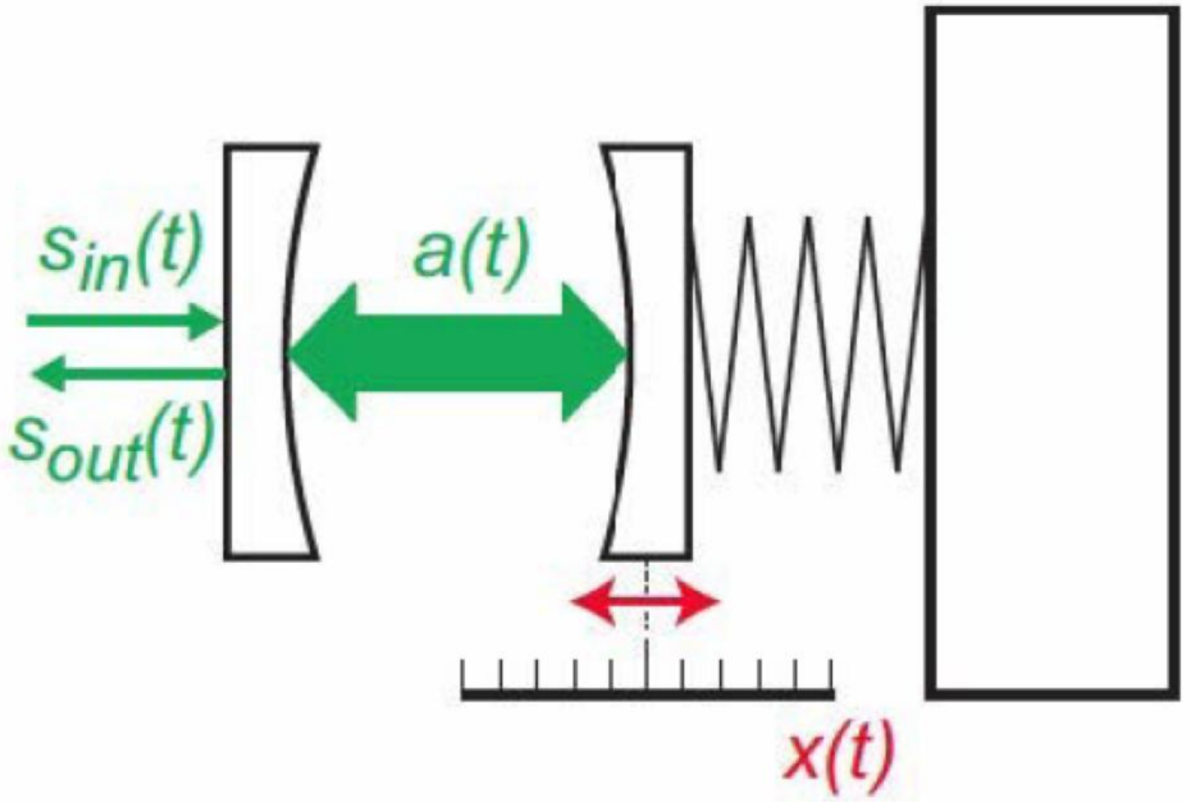
$$\hat{H}_\mathrm{int} = \hbar G x_\mathrm{zpf}\bar{a}(\delta\hat{b}\delta\hat{a}^\dagger + \delta\hat{b}^\dagger\delta\hat{a})$$

$$\hat{H}_\mathrm{int} = \hbar\frac{\Omega_\mathrm{c}}{2}(\delta\hat{b}\delta\hat{a}^\dagger + \delta\hat{b}^\dagger\delta\hat{a})$$

$$\Omega_\mathrm{c} = 2g_0\sqrt{\bar{n}_\mathrm{p}}$$

Corresponds to state swapping between optical and mechanical mode

Hamiltonian description



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Resolved sideband regime: $\Delta = -\Omega_\mathrm{m}$

Corresponds to state swapping between optical and mechanical mode

Linearized equations of motion

$$\begin{aligned}\delta\dot{a}_{\mathrm{p}} &= (i\Delta-\frac{\kappa}{2})\delta a_{\mathrm{p}}+ig_0\sqrt{\bar{n}_{\mathrm{cav}}}(\hat{b}+\hat{b}^{\dagger})+\sqrt{\kappa}\delta\hat{a}_{\mathrm{in}}(t)\\ \dot{b} &= (-i\Omega_{\mathrm{m}}-\frac{\Gamma_{\mathrm{m}}}{2})b+ig_0\sqrt{\bar{n}_{\mathrm{cav}}}(\delta\hat{a}+\delta\hat{a}^{\dagger})+\sqrt{\Gamma_{\mathrm{m}}}\delta\hat{b}_{\mathrm{in}}(t)\end{aligned}$$

Solutions to the coupled mode equations

$$\begin{aligned}\chi^{-1}(\omega) &= \Omega_{\mathrm{m}}^2+2\Omega_{\mathrm{m}}\Omega_{\mathrm{s}}(\omega)-\omega^2-i\omega[\Gamma_{\mathrm{m}}+\Gamma_{\mathrm{opt}}(\omega)]\\ \Omega_{\mathrm{eff}}(\omega) &= (\sqrt{\bar{n}_{\mathrm{cav}}}g_0)^2[\frac{\Delta+\omega}{(\Delta+\omega)^2+\kappa^2/4}+\frac{\Delta-\omega}{(\Delta-\omega)^2+\kappa^2/4}]\\ \Gamma_{\mathrm{opt}}(\omega) &= \frac{(\sqrt{\bar{n}_{\mathrm{cav}}}g_0)^2}{\omega}[\frac{\kappa}{(\Delta+\omega)^2+\kappa^2/4}+\frac{\kappa}{(\Delta-\omega)^2+\kappa^2/4}]\end{aligned}$$

Key approximation: $\omega=\Omega_{\mathrm{m}}, \kappa\gg\Gamma_{\mathrm{m}}$

Wilson-Rae, I., Nooshi, N., Zwerger, W. & Kippenberg, T. Theory of Ground State Cooling of a 0 Mechanical Oscillator Using Dynamical Backaction. Physical Review Letters 99, doi:10.1103/PhysRevLett.99.093901 (2007).
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$$\Delta = \omega_L - \omega_{cav}$$

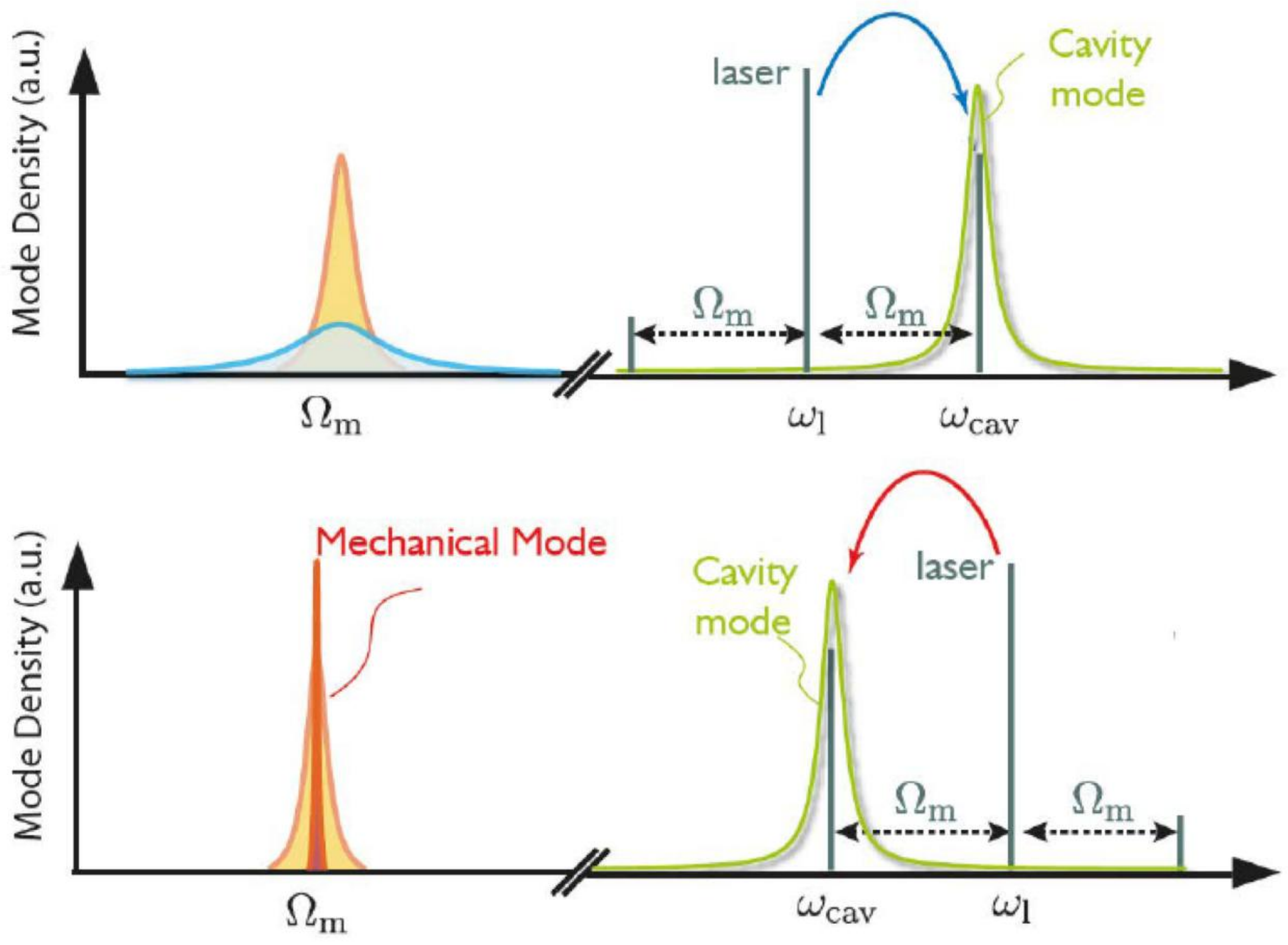
Liniearize: $\hat{a} = \bar{\alpha} + \delta\hat{a}$

Keep terms at least order of α and employ RWA

$\Delta = -\Omega_m$ $\hat{H}_{int} \approx -\hbar g_0\sqrt{\bar{n}_{cav}}(\hat{b}\delta\hat{a}^\dagger + \hat{b}^\dagger\delta\hat{a})$
Coherent exchange of quanta, cooling

$\Delta = +\Omega_m$ $\hat{H}_{int} \approx -\hbar g_0\sqrt{\bar{n}_{cav}}(\hat{b}^\dagger\delta\hat{a}^\dagger + \hat{b}\delta\hat{a})$
Two-mode squeezing, amplification

$\Delta = 0$ $\hat{H}_{int} \approx -\hbar g_0\sqrt{\bar{n}_{cav}}(\delta\hat{a} + \delta\hat{a}^\dagger)(\hat{b}+\hat{b}^\dagger)$
Sensitive readout of mechanical motion



Marquardt et al. Phys. Rev. Lett. 2007
Wilson Rae, Zwerger, Kippenberg Phys. Rev. Lett. 2007

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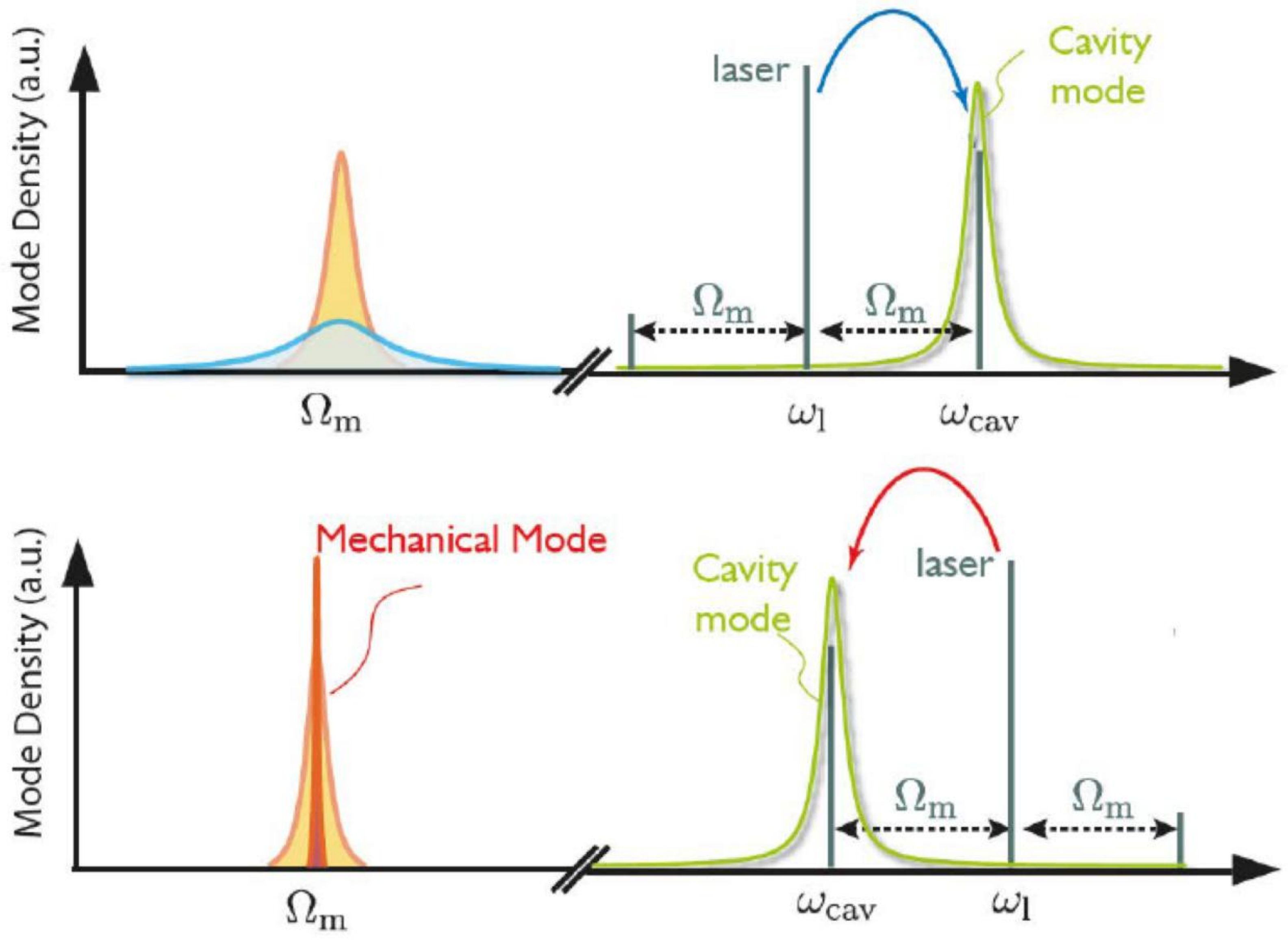

Marquardt et al. Phys. Rev. Lett. 2007
Wilson Rae, Zwerger, Kippenberg Phys. Rev. Lett. 2007

Quantum Langevin equations

Recall: Coupling between system and bath

$$H_{\mathrm{B}} = \int d\omega \hbar\omega \hat{b}^{\dagger}_{\omega}\hat{b}_{\omega} \quad \text{and} \quad H_{\mathrm{SB}} = \int d\omega g(\omega)(\hat{a}\hat{b}^{\dagger}_{\omega} + \hat{a}^{\dagger}\hat{b}_{\omega}).$$

Time-evolution of the operators in the Heisenberg picture gives a dissipation term and a fluctuation term:

$$\frac{d\hat{a}}{dt} = \frac{i}{\hbar}[\hat{a}, \hat{H}_{\mathrm{sys}}] - \frac{\kappa}{2}\hat{a} + \sqrt{\kappa}\hat{a}_{\mathrm{in}}.$$

For the optomechanical Hamiltonian, $\hat{H} = \hbar\omega_{\mathrm{c}}(1 - \frac{\hat{x}}{L})\hat{a}^{\dagger}\hat{a} + \hbar\Omega_{\mathrm{m}}\hat{b}^{\dagger}\hat{b}$, we get the following equations of motion:

$$\frac{d\hat{a}}{dt} = -i\omega_{\mathrm{c}}\hat{a} + i\frac{\omega_{\mathrm{c}}}{L}x_{\mathrm{zpf}}(\hat{b}^{\dagger} + \hat{b})\hat{a} - \frac{\kappa}{2}\hat{a} + \sqrt{\kappa}(\hat{a}_{\mathrm{in}} + \alpha e^{i\omega_{\mathrm{L}}t}),$$

$$\frac{d\hat{b}}{dt} = -i\Omega_{\mathrm{m}}\hat{b} + i\frac{\omega_{\mathrm{c}}}{L}\hat{a}^{\dagger}\hat{a} - \frac{\Gamma_{\mathrm{m}}}{2}\hat{b} + \sqrt{\Gamma_{\mathrm{m}}}\hat{b}_{\mathrm{in}}.$$

Quantum Langevin equations

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Quantum Langevin equations

We transfer to a rotating frame, $\hat{a} \to \hat{a}e^{i\omega_\mathrm{L}t}$. We consider the case when the cavity is resonantly driven, i.e. $\omega_\mathrm{L} = \omega_\mathrm{c}$. Next we assume that the fields are strong, so they can be represented as a sum of some mean value and small fluctuations:

$$\hat{a} \to \bar{\alpha} + \delta\hat{a} \quad \text{and} \quad \hat{b} \to \bar{\beta} + \delta\hat{b}.$$

The interaction Hamiltonian $\hat{H}_\mathrm{int} = \hbar\frac{\omega_\mathrm{c}}{L}x_\mathrm{zpf}(\hat{b}^\dagger + \hat{b})\hat{a}^\dagger\hat{a}$ is thus *linearized*:

$$\hat{a}^\dagger\hat{a} = (\bar{\alpha}^* + \delta\hat{a}^\dagger)(\bar{\alpha} + \delta\hat{a}) \to \bar{\alpha}(\delta\hat{a}^\dagger + \delta\hat{a})$$

Redefining $\delta\hat{a}$ as $\hat{a}$, we get *linearised quantum Langevin equations*:

$$\frac{d\hat{a}}{dt} = i\frac{\omega_\mathrm{c}}{L}x_\mathrm{zpf}(\hat{b}^\dagger + \hat{b})\hat{a} - \frac{\kappa}{2}\hat{a} + \sqrt{\kappa}\hat{a}_\mathrm{in},$$
$$\frac{d\hat{b}}{dt} = -i\Omega_\mathrm{m}\hat{b} + i\frac{\omega_\mathrm{c}}{L}\bar{\alpha}(a + a^\dagger) - \frac{\Gamma_\mathrm{m}}{2}\hat{b} + \sqrt{\Gamma_\mathrm{m}}\hat{b}_\mathrm{in}.$$

Quantum Langevin equations

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Redefining $\delta\hat{a}$ as $\hat{a}$, we get *linearised quantum Langevin equations*:

Quadratures

Next we consider the fluctuations in amplitude and phase quadratures:

$$\hat{X} = \hat{a} + \hat{a}^\dagger \quad \text{and} \quad \hat{Y} = \hat{a} - \hat{a}^\dagger$$

. The Langevin equations for the optical field can be expressed as

$$\frac{d\hat{X}}{dt} = -\frac{\kappa}{2}\hat{X} + \sqrt{\kappa}\hat{X}_{\mathrm{in}}$$
$$\frac{d\hat{Y}}{dt} = 2ix_{\mathrm{zpf}}\frac{\omega_{\mathrm{c}}}{L}\bar{\alpha}(\hat{b} + \hat{b}^\dagger) - \frac{\kappa}{2}\hat{Y} + \sqrt{\kappa}\hat{Y}_{\mathrm{in}}$$

Position of the mechanical oscillator is imprinted on the phase of the optical field $\implies$ We can infer position by using of *homodyne detection*.

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Input-output relations

The input-output relation for fields also applies to the quadratures:

$$\hat{a}_{\mathrm{out}} = -\hat{a}_{\mathrm{in}} + \sqrt{\kappa}\hat{a} \quad \Longrightarrow \quad \hat{Y}_{\mathrm{out}} = -\hat{Y}_{\mathrm{in}} + \sqrt{\kappa}\hat{Y}\,.$$

Taking the Fourier transform $\hat{Y}(\omega) = \int_{-\infty}^{\infty} e^{i\omega t}\hat{Y}(t)dt$, and defining $\hat{q} = x_{\mathrm{zpf}}(\hat{b}^\dagger + \hat{b})$, we have

$$-i\omega\hat{Y}(\omega) = 2i\frac{\omega_{\mathrm{c}}}{L}\bar{\alpha}\hat{q}(\omega) - \frac{\kappa}{2}\hat{Y}(\omega) + \sqrt{\kappa}\hat{Y}_{\mathrm{in}}(\omega).$$

After substitution, we assume so-called bad-cavity limit $\kappa \gg \omega$ and derive the output phase quadrature:

$$\hat{Y}_{\mathrm{out}}(\omega) = -\hat{Y}_{\mathrm{in}}(\omega) + i\frac{\bar{\alpha}\omega_{\mathrm{c}}}{L}\sqrt{\frac{8}{\kappa}}\hat{q}(\omega).$$

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After substitution, we assume so-called bad-cavity limit $\kappa \gg \omega$ and derive the output phase quadrature:

Spectral densities

We can find spectral densities either by definition or Wiener-Khinchin theorem:

$$S_{\hat{\mathrm{Y}}\hat{\mathrm{Y}}}(\omega) = \lim_{\tau\to\infty} \frac{1}{\tau}\langle \hat{Y}_\tau^\dagger(\omega)\hat{Y}_\tau(\omega)\rangle$$
$$= \int d\tau e^{-i\omega\tau}\langle \hat{Y}^\dagger(\tau)\hat{Y}(0)\rangle = \int_\infty^\infty d\omega' \langle \hat{Y}^\dagger(-\omega)\hat{Y}(\omega')\rangle$$

Spectral density of the output noise is given by

$$S_{\hat{\mathrm{Y}}_{\mathrm{out}}\hat{\mathrm{Y}}_{\mathrm{out}}}(\omega) = \underbrace{1}_{\text{Shot noise } \mathrm{S}_{\hat{\mathrm{Y}}_{\mathrm{in}}\hat{\mathrm{Y}}_{\mathrm{in}}}} + \underbrace{\frac{8\omega_{\mathrm{c}}^2\bar{\alpha}^2}{\kappa L^2} S_{\hat{\mathrm{q}}\hat{\mathrm{q}}}(\omega)}_{\text{signal}}$$

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Spectral density of the output noise is given by

SQL and Heisenberg uncertainty

The weakest signal $S^{imp}_{\hat{q}\hat{q}}$ can be measured when signal-to-noise ratio is equal to 1:

$$S^{imp}_{\hat{q}\hat{q}} = \left(\frac{\kappa L^2}{8\omega_c^2\bar{\alpha}^2}\right) S_{\hat{Y}_{in}\hat{Y}_{in}}$$

Force acting on the mechanical oscillator is $F = \partial \hat{H}/\partial \hat{q}$. Assuming $\dot{\hat{X}} = 0$,

$$\hat{F} = \sqrt{2}\hbar\frac{\omega_c}{L}\hat{X} \implies \hat{F} = \sqrt{\frac{8}{\kappa}}\hbar\frac{\omega_c}{L}\hat{X}_{in}$$

$$\implies S_{\hat{F}\hat{F}}(\omega) = \frac{8}{\kappa}\left(\hbar\frac{\omega_c}{L}\bar{\alpha}\right)^2 S_{\hat{X}_{in}\hat{X}_{in}}(\omega)$$

From these two expressions, it can be seen that

$$S_{\hat{F}\hat{F}}(\omega) S^{imp}_{\hat{q}\hat{q}} = \hbar^2 S_{\hat{Y}_{in}Y_{in}} S_{\hat{X}_{in}\hat{X}_{in}} = \frac{\hbar^2}{4}.$$

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Spectrum of position fluctuations

We write a second order differential equation for position:

$$\ddot{\hat{q}} = -\Omega_{\mathrm{m}}^2 \hat{q} - 2i\frac{\omega_{\mathrm{c}}}{L}\bar{\alpha} x_{\mathrm{zpf}} \hat{X} - \Gamma_{\mathrm{m}} \dot{\hat{q}} + \sqrt{\Gamma_{\mathrm{m}}} \hat{q}_{\mathrm{in}}.$$

Taking the Fourier transform as $\hat{q}(\omega) = \int_{-\infty}^{\infty} e^{i\omega t} \hat{q}(t) dt$, we get:

$$\hat{q}(\omega) = \chi(\omega) \left[-2i\frac{\omega_{\mathrm{c}}}{L}\bar{\alpha} x_{\mathrm{zpf}} \hat{X}(\omega) + \sqrt{\Gamma_{\mathrm{m}}} \hat{q}_{\mathrm{in}} \right],$$

where $\chi(\omega) = (\Omega_{\mathrm{m}}^2 - \omega^2 - i\omega\Gamma_{\mathrm{m}})^{-1}$.

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Taking the Fourier transform

Noise Spectral Density

$$S_{\hat{q}\hat{q}}(\omega) = 2\Gamma_{\mathrm{m}}|\chi(\omega)|^2 \left[S_{\hat{q}_{\mathrm{in}}\hat{q}_{\mathrm{in}}} + \underbrace{4\frac{\left(x_{\mathrm{zpf}}\bar{\alpha}\frac{\omega_{\mathrm{c}}}{L}\right)^2}{\Gamma_{\mathrm{m}}}}_{\mathrm{C_{eff}}} S_{\hat{\mathrm{X}}\hat{\mathrm{X}}} \right]$$

$$S_{\hat{q}\hat{q}}(\omega) = 2\Gamma_{\mathrm{m}}|\chi(\omega)|^2 (n_{\mathrm{th}} + C_{\mathrm{eff}} + 1)$$

$$S_{\hat{q}\hat{q}}(-\omega) = 2\Gamma_{\mathrm{m}}|\chi(\omega)|^2 (n_{\mathrm{th}} + C_{\mathrm{eff}})$$

Asymmetric noise spectral density $\rightarrow$ In total disagreement with classical results!

Noise Spectral Density

Asymmetric noise spectral density $\rightarrow$ In total disagreement with classical results!

Summary

In this lecture, you have seen:

- The physical origin of the optomechanical coupling
- Hamiltonian of an optomechanical system
- Optomechanical equations of motion
- Spectrum of position fluctuations